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Stabilizability of Systems with Two Point Delays by Using a Generalized PD Controller

C. F. ALASTRUEY
Department of Automatic Control and Computing Systems
Public University of Navarra (UPNA-NUP)
Arrosadiako Campusa, 31006 Iruña-Pamplona, Spain
karlos@upna.es

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Abstract—A generalization of a class of PD controllers is developed to stabilize a class of systems with two point delays in their state. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND PROBLEM STATEMENT

The stability for a scalar differential system with two point delays in its state has been considered in different works (see, for instance, [1]). However, the exact delay-dependent algebraic stability conditions of such a system were recently developed by Schoen and Geering [2] by using an instability criterion together with the D-decomposition method. That result has been extended to the MIMO case by Alastruey *et al.* [3]. In this note, a method is provided in order to design stabilizing generalized PD controllers for MIMO systems with two point delays in the state.

The rest of the paper is organized as follows. Section 2 introduces some preliminary results on the stability for a system with two point delays in its state that will be used in the sequel. Section 3 shows the main stabilizability result by using a generalized version of a class of PD controllers for systems with two point delays in the state. Finally, conclusions end the paper.

2. STABILITY FOR A SYSTEM WITH TWO POINT DELAYS

The following theorem will be useful in the next section in order to deduce stabilizing PD controllers.

THEOREM 2.1. *(See [3].) Consider the following linear MIMO system with two point delays in its vector-state:*

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + A_2x(t-2h) \tag{1}$$

with initial condition $\varphi_x : [-2h, 0] \to R^n$; $\varphi_x(0) = x(0)$ being absolutely continuous with possible bounded discontinuities on a subset of $[-2h, 0]$ of zero measure, and where A_0, A_1, and A_2 are

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real constant $n \times n$-matrices and $h > 0$. Consider the following $n \times n$-matrices:

$$A_k = A_{kd} + \bar{A}_{kd}, \qquad k = 0,1,2, \tag{2}$$
$$D = \operatorname{diag}\left[-\left(\left|a_{ii}^{2d}\right| - \frac{\pi}{2h}\right)\right], \tag{3}$$
$$E = -(A_{0d} + A_{1d} + A_{2d}), \tag{4}$$
$$F = \operatorname{diag}\left[\frac{y_i \cos(y_i h)}{\sin(y_i h)}\right], \tag{5}$$
$$G = \operatorname{diag}\left[\frac{y_i}{\sin(y_i h)} + 2a_{ii}^{2d}\cos(y_i h)\right], \tag{6}$$

where A_0, A_1, A_2 are decomposed according to (2) where $A_{kd} = \operatorname{diag}(a_{ii}^{kd})$, $(k = 0,1,2)$ are diagonal matrices, and $\bar{A}_{kd} = (a_{ij}^k,\ i \neq j$, otherwise $0) = (\bar{a}_{ij}^k) = A_k - A_{kd}$ are matrices with zero entries in their main diagonal; a_{ii}^{2d} are the elements of the diagonal matrix A_{2d} and $y_i \in [0, \pi/h]$, $(i = 1,\dots,n)$ are real numbers. Then the time-delay system (1) with D positive is asymptotically stable if the following four conditions hold for some set of values $y_i \in [0,\pi/h]$, $(i = 1,\dots,n)$:

(i) *E is positive;* (7)

(ii) *$A_{0d} = F + A_{2d}$;* (8)

(iii) *$A_{1d} + G$ is positive;* (9)

(iv)
$$\sum_{k=0}^{2}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left|a_{ij}^k\right|^2\right)^{1/2} < \max\left\{\left|\left(\sum_{i=1}^{n}\left|a_{ii}^0\right|^2\right)^{1/2} - \left(\sum_{i=1}^{n}\left|a_{ii}^1\right|^2\right)^{1/2} - \left(\sum_{i=1}^{n}\left|a_{ii}^2\right|^2\right)^{1/2}\right|;\right.$$
$$\left|\left(\sum_{i=1}^{n}\left|a_{ii}^1\right|^2\right)^{1/2} - \left(\sum_{i=1}^{n}\left|a_{ii}^0\right|^2\right)^{1/2} - \left(\sum_{i=1}^{n}\left|a_{ii}^2\right|^2\right)^{1/2}\right|; \tag{10}$$
$$\left.\left|\left(\sum_{i=1}^{n}\left|a_{ii}^2\right|^2\right)^{1/2} - \left(\sum_{i=1}^{n}\left|a_{ii}^0\right|^2\right)^{1/2} - \left(\sum_{i=1}^{n}\left|a_{ii}^1\right|^2\right)^{1/2}\right|\right\}.$$

3. MAIN STABILIZABILITY RESULT FOR A SYSTEM WITH TWO POINT DELAYS

In this section, conditions for a generalized PD control law to stabilize asymptotically a system with two point delays in its state will be discussed. Consider the following MIMO system with two point delays in its state, and a control law:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) + Bu(t),$$
$$y(t) = Cx(t), \tag{11a}$$
$$x(t) = \varphi(t), \qquad \forall\, t \in [-2h, 0); \quad x(0) = x_0, \tag{11b}$$

where $x(t), y(t)$ represent, respectively, the state and the output of the system. Additionally, $A_i, B, C \in W \subset R^{n\times n}$, $i = 0,1,2$, where W is the class of $n \times n$ real matrices Q such that $\|Q\| < \infty$. Hence it is deduced that $x(t)$, $u(t)$, and $y(t)$ are $n \times n$-vectors.

From a traditional viewpoint, a standard PD controller could be defined for system (11) as $u(t) = K_0 y(t) + K_d \dot{y}(t)$, where K_0, K_d are $n \times n$-matrices representing the direct (proportional) and indirect (derivative) control, respectively. However, a generalized PD controller is now introduced.

DEFINITION 3.1. *A generalized PD controller for system (11) is defined as*

$$\begin{aligned} u(t) &= K_0 y(t) + K_1 y(t-h) + K_2 y(t-2h) + K_d \dot{y}(t), \\ &\text{where } K_i, K_d \in W \subset R^{n\times n}, \qquad i = 0,1,2. \end{aligned} \tag{12}$$

Let us introduce some auxiliary matrices before presenting the main result. Define the following $n \times n$ real matrices, based on the matrices appearing in the plant under study (11) and the generalized PD controller (12):

$$\hat{A}_i = \left[I - BK_dC\right]^{-1}\left[A_k + BK_kC\right], \qquad k = 0,1,2. \tag{13}$$

Decompose

$$\hat{A}_k = \hat{A}_{kd} + \overline{\hat{A}}_{kd}, \qquad k = 0,1,2, \tag{14}$$

$$\hat{D} = \operatorname{diag}\left[-\left(\left|\hat{a}_{ii}^{2d}\right| - \frac{\pi}{2h}\right)\right], \tag{15}$$

$$\hat{E} = -\left(\hat{A}_{0d} + \hat{A}_{1d} + \hat{A}_{2d}\right), \tag{16}$$

$$\hat{F} = \operatorname{diag}\left[\frac{y_i \cos(y_i h)}{\sin(y_i h)}\right], \tag{17}$$

$$\hat{G} = \operatorname{diag}\left[\frac{y_i}{\sin(y_i h)} + 2\hat{a}_{ii}^{2d}\cos(y_i h)\right], \tag{18}$$

where $\hat{A}_0, \hat{A}_1, \hat{A}_2$ are decomposed according to (14) where $\hat{A}_{kd} = \operatorname{diag}(\hat{a}_{ii}^{kd})$, $(k = 0,1,2)$ are diagonal matrices, and $\overline{\hat{A}}_{kd} = (\hat{a}_{ij}^k,\ i \neq j, \text{ otherwise } 0) = (\overline{\hat{a}}_{ij}^k) = \hat{A}_k - \hat{A}_{kd}$ are matrices with zero entries in their main diagonal; $\hat{a}_{ii}^{2d}$ are the elements of the diagonal matrix $\hat{A}_{2d}$ and $y_i \in [0, \pi/h]$, $(i = 1, \dots, n)$ are real numbers. Now the main result can be established.

THEOREM 3.1. *A generalized PD controller (12) provides asymptotic stability for system (11) if the following six conditions hold for some set of values $y_i \in [0, \pi/h]$, $(i = 1, \dots, n)$:*

(i) $\hat{D}$ *is positive;* (19)

(ii) $\hat{E}$ *is positive;* (20)

(iii) $\hat{A}_{0d} = \hat{F} + \hat{A}_{2d}$; (21)

(iv) $\hat{A}_{1d} + \hat{G}$ *is positive;* (22)

(v)
$$\begin{aligned} \sum_{k=0}^{2}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left|\hat{a}_{ij}^k\right|^2\right)^{1/2} < \max\Bigg\{ &\left|\left(\sum_{i=1}^n\left|\hat{a}_{ii}^0\right|^2\right)^{1/2} - \left(\sum_{i=1}^n\left|\hat{a}_{ii}^1\right|^2\right)^{1/2} - \left(\sum_{i=1}^n\left|\hat{a}_{ii}^2\right|^2\right)^{1/2}\right|; \\ &\left|\left(\sum_{i=1}^n\left|\hat{a}_{ii}^1\right|^2\right)^{1/2} - \left(\sum_{i=1}^n\left|\hat{a}_{ii}^0\right|^2\right)^{1/2} - \left(\sum_{i=1}^n\left|\hat{a}_{ii}^2\right|^2\right)^{1/2}\right|; \\ &\left|\left(\sum_{i=1}^n\left|\hat{a}_{ii}^2\right|^2\right)^{1/2} - \left(\sum_{i=1}^n\left|\hat{a}_{ii}^0\right|^2\right)^{1/2} - \left(\sum_{i=1}^n\left|\hat{a}_{ii}^1\right|^2\right)^{1/2}\right|\Bigg\}; \end{aligned} \tag{23}$$

(vi) $I - BK_dC$ *is nonsingular.* (24)

PROOF. See that system (11) with a generalized PD controller (12) in closed-loop form can be written as

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + A_2x(t-2h) \\ &\quad + BK_0y(t) + BK_1y(t-h) + BK_2y(t-2h) + BK_d\dot{y}(t) \\ &= A_0x(t) + A_1x(t-h) + A_2x(t-2h) \\ &\quad + BK_0Cx(t) + BK_1Cx(t-h) + BK_2Cx(t-2h) + BK_dC\dot{x}(t). \end{aligned} \tag{25}$$

Equation (25) can be rewritten as

$$[I - BK_dC]\dot{x}(t) = [A_0 + BK_0C]x(t) + [A_1 + BK_1C]x(t-h) + [A_2 + BK_2C]x(t-2h). \tag{26}$$

By using assumption (vi) one obtains

$$\begin{aligned}\dot{x}(t) &= [I - BK_dC]^{-1}[A_0 + BK_0C]x(t) + [I - BK_dC]^{-1}[A_1 + BK_1C]x(t-h) \\ &\quad + [I - BK_dC]^{-1}[A_2 + BK_2C]x(t-2h),\end{aligned} \tag{27}$$

and finally, by using equation (13), the closed-loop system with a generalized PD controller can be written as

$$\dot{x}(t) = \hat{A}_0 x(t) + \hat{A}_1 x(t-h) + \hat{A}_2 x(t-2h), \tag{28}$$

but, by hypothesis, system (28) verifies conditions (ii)–(v) and precondition (i) in Theorem 3.1; then, by Theorem 2.1, system (28) is asymptotically stable; therefore system (11) is asymptotically stabilizable by a generalized PD controller (12) subject to assumptions (19)–(24).

4. CONCLUSIONS

This note provides sufficient conditions for providing asymptotic stabilizability of a class of delay-differential systems with two point delays in the state, by using a generalized PD controller.

Conditions are given in terms of algebraic relations, which are very suitable for computer applications.

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